Relating fuzzy autoepistemic logic to fuzzy modal logics of belief

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Abstract

Fuzzy autoepistemic logic is a generalization of autoepistemic logic, an important formalism for nonmonotonic reasoning originally intended to model an ideally rational agent reflecting upon his own beliefs, and allows to represent an agent's rational beliefs on partially true gradable propositions. Fuzzy autoepistemic logic has recently been shown to be a suitable logical framework for fuzzy answer set programming, generalizing a classical result. On the other hand, there are well-known links between autoepistemic logic and several nonmonotonic modal logic systems. In this paper, we introduce generalizations of the main classical propositional modal logics of belief based on finitely-valued Łukasiewicz calculus. We obtain completeness with respect to appropriate Kripke-style semantics and we prove NP-completeness for the satisfiability problem. Then we show how fuzzy autoepistemic logic can be approached in these many-valued modal settings. In particular we obtain a generalization of Levesque's result on the relationship between stable expansions, belief sets and "only knowing" operators.

1 Introduction and motivation

Since its introduction in the 1980s, autoepistemic logic [Moore, 1983; Konolige, 1994; Shvarts, 1990] has been one of the main formalisms for nonmonotonic reasoning. It extends propositional logic by offering the ability to reason about an agent's (lack of) beliefs. More precisely, these beliefs are sets of sentences in a propositional language augmented by a modal operator B. If φ is a formula, then $\mathrm{B}\varphi$, which has to be interpreted as "φ is believed", is a formula as well. Hence, in this language nested modal operators are allowed; it is possible to have beliefs about beliefs.

Logic programming has had a significant impact on the development of nonmonotonic logics and vice versa (e.g. [Baral and Gelfond, 1994]). In particular, Gelfond and Lifschitz [Gelfond and Lifschitz, 1988] showed that there is a one-to-one correspondence between the answer sets of an answer set program and the stable expansions of a corresponding autoepistemic theory. In [Moore, 1983], a *stable expansion* of a set of autoepistemic formulas A is defined as a set of formulas E_A such that the following fix-point condition holds:

$$E_A = \{\varphi \mid A \cup \{\mathrm{B}\psi \mid \psi \in E_A\} \cup \{\neg\mathrm{B}\psi \mid \psi \notin E_A\} \vdash \varphi\},$$

where $\vdash$ denotes derivability in classical propositional logic and each formula $\mathrm{B}\varphi$ is considered as a new propositional variable. Informally, a stable expansion of A is a closed set of beliefs of an ideal rational agent based on the premises A.

In [Levesque, 1990], a modal logic account of main concepts of Moore's autoepistemic logic is provided, and in particular a K45 modal logic of belief is expanded with a new unary modality O where $\mathrm{O}\varphi$ has to be interpreted as "φ is all that is believed" or "only φ is believed". It is then shown that "only believing" (or "only knowing") is closely related to stable expansions in autoepistemic logic.

Recently, a fuzzy generalization (from a semantical point of view) of autoepistemic logic has been defined in [Blondeel *et al.*, 2013] where it is also shown that the important relation between autoepistemic logic and answer set programming is preserved: the answer sets of a fuzzy answer set program (e.g. [Van Nieuwenborgh *et al.*, 2007]) can be equivalently determined by computing the fuzzy stable expansions of a corresponding set of fuzzy autoepistemic formulas.

In this paper, we introduce generalizations of the main classical propositional modal logics of belief (K45, KD45, S5) based on finitely-valued Łukasiewicz calculus in order to model the notion of belief on fuzzy propositions, in the sense of admitting partial degrees of truth between 0 (fully false) and 1 (fully true). For instance suppose the expression "I believe it is raining" has truth value 0.2. This is interpreted as "I believe to degree 0.2 that it is raining", or as can be shown by the definitions we will introduce as "I (fully) believe that it is raining to at least degree 0.2". For practical and technical reasons we consider truth degrees belonging to a finite

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scale $S_k = \{0, \frac{1}{k}, \dots, \frac{k-1}{k}, 1\}$. Then we show how fuzzy autoepistemic logic can be approached using possible worlds semantics corresponding to these many-valued modal logics. We also consider the expansion of our many-valued K45 with an "only believing" operator O and show that Levesque's result on the relationship between stable expansions, belief sets and "only knowing" operators nicely extends to our framework.

This paper is structured as follows. After this introduction, in Section 2, we provide some necessary preliminaries on the $(k+1)$-valued Łukasiewicz logic $Ł_k$ and available results on the minimal modal logic over $Ł_k$. In Section 3 we define proper generalizations of the classical modal systems K45, KD45 and S5 and prove sound- and completeness with respect to appropriate Kripke-style semantics, while in Section 4 we deal with the complexity of these logics and prove NP-completeness for two variants of the satisfiability problem. Then in Section 5 we consider possible world semantics for the fuzzy autoepistemic logic of [Blondeel *et al.*, 2013] and provide a characterization of fuzzy stable expansions in terms of many-valued K45 belief sets, and also in terms of proper generalizations of stable sets. In Section 6, we generalize (from a semantical point of view) the propositional fragment of Levesque's "only knowing" logic and prove that there is a characterization of fuzzy stable expansions in terms of the belief sets involving the "only knowing" operator O. We conclude with some final remarks about related work.

2 Background

Consider the propositional language $\mathcal{L}$ whose formulas are built from a countable set of propositional variables V, the connective $\rightarrow$ (implication) and truth constants $\overline{c}$ for each $c \in S_k = \{0, \frac{1}{k}, \dots, \frac{k-1}{k}, 1\}$ with a fixed $k \in \mathbb{N}$. Further connectives are defined as follows:

$$\begin{array}{ll} \neg\phi = \phi \rightarrow \overline{0} & \phi \wedge \psi = \phi \otimes (\phi \rightarrow \psi) \\ \phi \otimes \psi = \neg(\phi \rightarrow \neg\psi) & \phi \oplus \psi = \neg(\neg\phi \otimes \neg\psi) \\ \phi \vee \psi = ((\phi \rightarrow \psi) \rightarrow \psi) & \phi \leftrightarrow \psi = (\phi \rightarrow \psi) \otimes (\psi \rightarrow \phi) \end{array}$$

with ϕ and ψ arbitrary formulas. A *propositional evaluation* is a mapping $e : V \rightarrow S_k$ that is extended to formulas as follows. If ϕ and ψ are formulas and c is an element in S_k, then

$$e(\phi \rightarrow \psi) = e(\phi) \Rightarrow e(\psi) \text{ and } e(\overline{c}) = c,$$

where $x \Rightarrow y = \min(1, 1 - x + y)$ for $x, y \in S_k$. Note that $(x \Rightarrow y) = 1$ iff $x \leq y$. The set of all such evaluations will be henceforth denoted by Ω_k. Notice that, in particular, for every formula ϕ and ψ and for every $e \in \Omega_k$, we have $e(\neg\phi) = 1 - e(\phi)$, $e(\phi \otimes \psi) = \max(e(\phi) + e(\psi) - 1, 0)$, $e(\phi \vee \psi) = \max(e(\phi), e(\psi))$, $e(\phi \wedge \psi) = \min(e(\phi), e(\psi))$, $e(\phi \oplus \psi) = \min(1, e(\phi) + e(\psi))$ and $e(\phi \leftrightarrow \psi) = 1 - |(e(\phi) - e(\psi)|$.

A formula ϕ is said to be *satisfiable* if there is some propositional evaluation e such that $e(\phi) = 1$. In such a case we say that e is a *model* of ϕ. A *tautology* is a formula ϕ such that $e(\phi) = 1$ for each propositional evaluation e. A formula ϕ is a *semantic consequence* of a set of formulas Γ, written as $\Gamma \models \phi$ iff it holds that if e is a model of each formula in Γ, then e is also a model of ϕ. The logic $Ł_k$ based on the language $\mathcal{L}$ has a sound and a strongly complete axiomatization, see e.g. [Cignoli *et al.*, 2000] for details. So if $\vdash$ denotes the notion of proof defined from the set of axioms of $Ł_k$ and modus ponens, then for any, hence possibly infinite, set of formulas $T \cup \{\psi\}$, it holds that $T \vdash \psi$ iff $T \models \psi$. A formula ψ that can be proven from axioms and rules only is called a *theorem* and this is written as $\vdash \psi$.

To reason with beliefs over fuzzy propositions we introduce a modal operator B. By $\mathcal{L}_{\mathcal{B}}$ we denote the expansion of $\mathcal{L}$ by B. We base our approach on previous theoretical work [Bou *et al.*, 2011b] on fuzzy modal logics where the truth-values form a finite residuated lattice. In [Bou *et al.*, 2011b], the authors introduce the minimal modal logic over $Ł_k$. Its axioms are all the axioms of $Ł_k$, plus the following.

(B2) $(\mathrm{B}\varphi \wedge \mathrm{B}\psi) \rightarrow \mathrm{B}(\varphi \wedge \psi)$,

(B3) $\mathrm{B}(\overline{c} \rightarrow \varphi) \leftrightarrow (\overline{c} \rightarrow \mathrm{B}\varphi)$, for each $c \in S_k$,

(B4) $(\mathrm{B}\varphi \oplus \mathrm{B}\varphi) \leftrightarrow \mathrm{B}(\varphi \oplus \varphi)$.

The rules are modus ponens (from ϕ and $\phi \rightarrow \psi$ infer ψ) and monotonicity for B (if $\phi \rightarrow \psi$ is a theorem then $\mathrm{B}\phi \rightarrow \mathrm{B}\psi$ is a theorem as well). In [Bou *et al.*, 2011b], the authors show that this is a sound and complete axiomatization with respect to the class of *Kripke frames* $M = (W, e, R)$ where W is a set of possible worlds, $e : W \times V \rightarrow S_k$ is a mapping giving an evaluation $e(w, \cdot) : V \rightarrow S_k$ for each possible world w and $R : W \times W \rightarrow S_k$ is a S_k-valued binary relation on possible worlds. Given a (Kripke) frame $M = (W, e, R)$ and a world $w \in W$, the truth value of a formula ϕ in $\mathcal{L}_{\mathcal{B}}$ is inductively defined as follows:

- If ϕ is a propositional variable p, then $\|\phi\|_{M,w} = e(w, p)$.
- If ϕ is truth-constant $\overline{c}$, then $\|\phi\|_{M,w} = c$.
- If $\phi = \mathrm{B}\psi$, then $\|\phi\|_{M,w} = \inf\{R(w, w') \Rightarrow \|\psi\|_{M,w'} \mid w' \in W\}$.
- If $\phi = \psi \rightarrow \gamma$, then $\|\phi\|_{M,w} = \|\psi\|_{M,w} \Rightarrow \|\gamma\|_{M,w}$.

The third bullet then intuitively expresses that ψ is believed in a world $w \in W$ to the degree that ψ is true in all worlds w' that are accessible (related to) from w to a certain degree. A formula ϕ is said to be *satisfiable* if there exists a frame $M = (W, e, R)$ and a $w \in W$ such that $\|\phi\|_{M,w} = 1$ and we say that ϕ is *satisfied* by M. It is called *a tautology* if for each frame $M = (W, e, R)$ we have $\|\phi\|_{M,w} = 1$ for each $w \in W$. A formula ϕ is a *semantic consequence* of a set of formulas Γ, written as $\Gamma \models_{\mathrm{B}} \phi$ iff it holds that if each formula in Γ is satisfied by a frame M, that ϕ is also satisfied by M.

As it was shown in [Bou *et al.*, 2011b], the well-known axiom

(K) $\mathrm{B}(\phi \rightarrow \psi) \rightarrow (\mathrm{B}\phi \rightarrow \mathrm{B}\psi)$

is not generally sound in the above Kripke frames, only in frames $M = (W, e, R)$ where R is a two-valued relation on M (i.e. when $R(w, w') \in \{0, 1\}$ for all $w, w' \in W$). Notice that in such Kripke frames, the truth evaluation of $\mathrm{B}\phi$ in a world $w \in W$ reduces to

$$\|\mathrm{B}\phi\|_{M,w} = \inf\{\|\phi\|_{M,w'} \mid R(w, w') = 1\}.$$

In the remainder of the paper we will be interested in the class of Kripke frames with two-valued accessibility relations. We will denote this class by $\mathbb{M}$. Moreover we will denote by $\mathrm{BŁ}_k$ the axiomatic extension of the minimal modal logic with axiom (K). Due to the presence of axiom (K), the monotonicity rule can be replaced by the usual necessitation rule: if ϕ is a theorem then $\mathrm{B}\phi$ is a theorem as well.

3 Fuzzy modal logics of belief

When defining logics for belief, it is usual to presume that the agent has both positive and negative introspective capabilities. This is captured in the classical case by the well-known axioms (4) and (5). Moreover, sometimes belief consistence is required which is captured by axiom (D). Finally, when dealing with knowledge instead of beliefs, i.e. beliefs are true, modal axiom (T) can be added. As we will show, this can be generalized to our many-valued setting. Thus we will consider some extensions of $\mathrm{BŁ}_k$ obtained by combining the following classical modal axioms:

(K)	$\mathrm{B}(\phi \to \psi) \to (\mathrm{B}\phi \to \mathrm{B}\psi)$	(5)	$\neg\mathrm{B}\phi \to \mathrm{B}\neg\mathrm{B}\phi$
(D)	$\neg\mathrm{B}\neg\overline{1}$	(T)	$\mathrm{B}\phi \to \phi$
(4)	$\mathrm{B}\phi \to \mathrm{BB}\phi$		

As in the classical case [Chellas, 1980; Fagin *et al.*, 1994], we consider the following extensions of $\mathrm{BŁ}_k$

- $K45(Ł_k)$: $\mathrm{BŁ}_k$ plus axioms (4) and (5),
- $KD45(Ł_k)$: $\mathrm{BŁ}_k$ plus axioms (D), (4) and (5)
- $S5(Ł_k)$: $\mathrm{BŁ}_k$ plus axioms (T), (4) and (5)

We will denote by $\vdash_L$ the notion of proof for any of the logics $L \in \{K45(Ł_k), KD45(Ł_k), S5(Ł_k)\}$. The next task is to prove completeness of these logics[1] with respect to corresponding classes of many-valued Kripke frames.

As was done for (fuzzy) autoepistemic logic [Blondeel *et al.*, 2013; Moore, 1983], each formula ϕ in $\mathcal{L}_\mathrm{B}$ can be seen as a formula ϕ^* in $Ł_k$ by treating subformulas $\mathrm{B}\psi$ as new propositional variables. For instance, $\mathrm{B}(a \wedge \mathrm{B}b)$ is seen as a fresh variable $p_{\mathrm{B}(a\wedge \mathrm{B}b)}$ and has no connection to $\mathrm{B}b$. Explicitly, for a variable p, truth-constant $\overline{c}$, formulas ϕ and ψ we define: $p^* = p$, $\overline{c}^* = \overline{c}$, $(\phi \to \psi)^* = \phi^* \to \psi^*$ and $(\mathrm{B}\phi)^* = p_\phi$ with p_ϕ a new variable. As a first step, we show a relation between proving a formula ψ in one of the extensions of $\mathrm{BŁ}_k$ and proving the corresponding B-free formula ψ^* from a suitable theory over the propositional logic $Ł_k$. For a set of formulas A, we define $A^* = \{\psi^* \mid \psi \in A\}$.

Lemma 3.1. *Let L be any of the logics $K45(Ł_k)$, $KD45(Ł_k)$, $S5(Ł_k)$. Suppose $T \cup \{\psi\}$ is a set of formulas from $\mathcal{L}_B$ and let $\Lambda_L = \{\phi^* \mid \vdash_L \phi\}$. Then it holds that* $T \vdash_L \psi$ *iff* $T^* \cup \Lambda_L \vdash \psi^*$.

Proof. Suppose a proof for ψ in L from T has the form $\Gamma = (\gamma_1, \dots, \gamma_m)$. A proof for ψ^* in $Ł_k$ from $T^* \cup \Lambda_L$ is then easily obtained by replacing all formulas γ_i in Γ by γ_i^*.

Conversely, suppose there is a proof $\Phi = (\phi_1, \dots, \phi_n)$ for ψ^* in $Ł_k$ from $T^* \cup \Lambda_L$. For each i we then have that $\phi_i = \gamma_i^*$ with either $\gamma_i \in T$ or γ_i an instantiation of an axiom in L or $\gamma_i = \mathrm{B}\alpha$ with $\vdash_L \alpha$. The sequence $\Gamma = (\gamma_1, \dots, \gamma_n)$ obtained from Φ is then converted to a proof for ψ in L from T as follows. If for some i, $\gamma_i \notin T$ and γ_i is not an axiom in L, then add a proof for $\gamma_i = \mathrm{B}\alpha$, which is possible since in this case, it must hold that $\vdash_L \alpha$ from which we can then infer that $\gamma_i = \mathrm{B}\alpha$ is a theorem as well. □

The next step is to define the canonical Kripke frame for a given modal logic L. The following definition is general for any logic L resulting from all possible combinations of the axioms (D), (4), (5) and (T) and hence in particular for $L \in \{K45(Ł_k), KD45(Ł_k), S5(Ł_k)\}$. The L-canonical Kripke frame is defined as the Kripke frame $M^L_{can} = (W^L_{can}, e^L_{can}, R^L_{can})$, where

- $W^L_{can} = \{w \in \Omega_k \mid \forall\phi \in \Lambda_L : w(\phi^*) = 1\}$ with $\Lambda_L = \{\phi^* \mid \vdash_L \phi\}$,
- $R^L_{can} = \{(w_1, w_2) \in \Omega_k \times \Omega_k \mid \forall\phi :$ if $w_1((\mathrm{B}\phi)^*) = 1$, then $w_2(\phi^*) = 1\}$,
- $e^L_{can}(w, p) = w(p)$ for each variable p.

We now introduce some subclasses of $\mathbb{M}$, depending on which properties the two-valued accessibility relations in the Kripke frames (W, e, R) satisfy.

- $\mathbb{M}_{\mathrm{et}}$: class of Kripke frames with Euclidean[2] and transitive relations
- $\mathbb{M}_{\mathrm{est}}$: class of Kripke frames with Euclidean, serial and transitive relations
- $\mathbb{M}_{\mathrm{rst}}$: class of Kripke frames with reflexive, symmetric and transitive relations

In Theorem 3.4 we will show that the extensions of $\mathrm{BŁ}_k$ defined above are sound and complete axiomatizations for these subclasses of $\mathbb{M}$. To show completeness, we need to prove the following truth lemma (general for any L).

Proposition 3.2. *(Truth-lemma) For any $\mathcal{L}_\mathcal{B}$-formula ϕ, let $M^L_{can} = (W^L_{can}, e^L_{can}, R^L_{can})$ be its canonical Kripke frame with $L \in \{K45(Ł_k), KD45(Ł_k), S5(Ł_k)\}$. Then it holds that $v(\phi^*) = \|\phi\|_{M^L_{can},v}$, for every $v \in W^L_{can}$.*

Proof. By using the monotonicity for B and the meet distribution property, the claim follows by an easy adapation from Lemma 4.20 in [Bou *et al.*, 2011b]. □

We can now show the following properties for the canonical Kripke frames.

Proposition 3.3. *Let $L \in \{K45(Ł_k), KD45(Ł_k), S5(Ł_k)\}$, then the following conditions hold*

1. *If L contains axiom (T) then R^L_{can} is reflexive.*
2. *If L contains axiom (4) then R^L_{can} is transitive.*
3. *If L contains axiom (5) then R^L_{can} is Euclidean.*
4. *If L contains axiom (D) then R^L_{can} is serial.*

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[1] We restrict ourselves to the logics $K45(Ł_k)$, $KD45(Ł_k)$ and $S5(Ł_k)$, but completeness results could be obtained for any of the logics resulting from other combinations of the above axioms.

[2] Recall that a relation R is called Euclidean if $R(w_1, w_2) = R(w_1, w_3) = 1$ implies $R(w_2, w_3) = 1$ for each $w_1, w_2, w_3 \in W_{can}$ and serial if for each $w_1 \in W$ there exists $w_2 \in W$ such that $R(w_1, w_2) = 1$.

Proof. **1**. Let $w \in W^L_{can}$ and $w((\mathrm{B}\phi)^*) = 1$. Since $(\mathrm{B}\phi \rightarrow \phi)^* \in \Lambda_L$, it follows that $1 = w((\mathrm{B}\phi \rightarrow \phi)^*) = \|\mathrm{B}\phi \rightarrow \phi\|_{M^L_{can},w}$ and hence that $1 = w((\mathrm{B}\phi)^*) = \|\mathrm{B}\phi\|_{M^L_{can},w} \leq \|\phi\|_{M^L_{can},w} = w(\phi^*)$. Therefore $R^L_{can}(w, w) = 1$.

2. Let $w_1, w_2, w_3 \in W^L_{can}$ such that $R^L_{can}(w_1, w_2) = R^L_{can}(w_2, w_3) = 1$ and $w_1((\mathrm{B}\phi)^*) = 1$. Since $(\mathrm{B}\phi \rightarrow \mathrm{BB}\phi)^* \in \Lambda_L$, it follows that $1 = w_1((\mathrm{B}\phi \rightarrow \mathrm{BB}\phi)^*) = \|\mathrm{B}\phi \rightarrow \mathrm{BB}\phi\|_{M^L_{can},w_1}$, and hence that $1 = w_1((\mathrm{B}\phi)^*) = \|\mathrm{B}\phi\|_{M^L_{can},w_1} \leq \|\mathrm{BB}\phi\|_{M^L_{can},w_1} = w_1((\mathrm{BB}\phi)^*)$. Since $R^L_{can}(w_1, w_2) = 1$, it then follows that $w_2((\mathrm{B}\phi)^*) = 1$ and subsequently, since $R^L_{can}(w_2, w_3) = 1$, that $w_3(\phi^*) = 1$. Hence $R^L_{can}(w_1, w_3) = 1$.

3. Let $w_1, w_2, w_3 \in W^L_{can}$ such that $R^L_{can}(w_1, w_2) = R^L_{can}(w_1, w_3) = 1$ and $w_2((\mathrm{B}\phi)^*) = 1$. By definition,

$$\|\mathrm{B}\neg\mathrm{B}\phi\|_{M^L_{can},w_1} = \inf\{\|\neg\mathrm{B}\phi\|_{M^L_{can},w} \mid R^L_{can}(w_1, w) = 1\},$$

hence $\|\mathrm{B}\neg\mathrm{B}\phi\|_{M^L_{can},w_1} \leq \|\neg\mathrm{B}\phi\|_{M^L_{can},w_2}$. Now since $\|\neg\mathrm{B}\phi\|_{M^L_{can},w_2} = 1 - \|\mathrm{B}\phi\|_{M^L_{can},w_2} = 1 - w_2((\mathrm{B}\phi)^*) = 0$, we obtain $\|\mathrm{B}\neg\mathrm{B}\phi\|_{M^L_{can},w_1} = 0$. But since $(\neg\mathrm{B}\neg\mathrm{B}\phi \rightarrow \mathrm{B}\phi)^* \in \Lambda_L$, it follows that

$$1 = w_1((\neg\mathrm{B}\neg\mathrm{B}\phi \rightarrow \mathrm{B}\phi)^*) = \|\neg\mathrm{B}\neg\mathrm{B}\phi \rightarrow \mathrm{B}\phi\|_{M^L_{can},w_1}$$

and hence $1 = \|\neg\mathrm{B}\neg\mathrm{B}\phi\|_{M^L_{can},w_1} \leq \|\mathrm{B}\phi\|_{M^L_{can},w_1} = w_1((\mathrm{B}\phi)^*)$. Finally, since $R^L_{can}(w_1, w_3) = 1$, it then follows that $w_3(\phi^*) = 1$, and hence $R^L_{can}(w_2, w_3) = 1$.

4. Let $w_1 \in W^L_{can}$. Since $(\neg\mathrm{B}\neg\overline{1})^* \in \Lambda_L$, it follows that $1 = w_1((\neg\mathrm{B}\neg\overline{1})^*) = \|\neg\mathrm{B}\neg\overline{1}\|_{M^L_{can},w_1}$, and thus $0 = \|\mathrm{B}\neg\overline{1}\|_{M^L_{can},w_1} = \inf\{\|\overline{0}\|_{M^L_{can},w} \mid R^L_{can}(w_1, w) = 1\}$. Therefore the latter set must be non-empty, and hence there must exist $w_2 \in W^L_{can}$ such that $R^L_{can}(w_1, w_2) = 1$. □

Using Proposition 3.3, we can now show the following theorem.

Theorem 3.4. *$K45(Ł_k)$, $KD45(Ł_k)$ and $S5(Ł_k)$ are sound and complete w.r.t. the classes $\mathbb{M}_{\text{et}}$, $\mathbb{M}_{\text{est}}$ and $\mathbb{M}_{\text{rst}}$ respectively.*

Proof. Soundness is straightforward. We can show the completeness by proving that if there is a formula ϕ such that $\nvdash_L \phi$ with $L \in \{K45(Ł_k), KD45(Ł_k), S5(Ł_k)\}$, then there must exist a Kripke frame $M = (W, e, R)$ in the corresponding subclass of Kripke frames such that there exists $w \in W$ with $\|\phi\|_{M,w} < 1$. We show that the L-canonical Kripke frame meets this condition. The fact that each of these canonical Kripke frames belong to the correct subclass of $\mathbb{M}$ follows from Proposition 3.3 and by the fact that a relation that is reflexive and Euclidean is also symmetrical. By Lemma 3.1 it follows, independently on L, that $\Lambda_L \nvdash \phi^*$ and by the strong completeness of $Ł_k$ it then follows that $\Lambda_L \nvDash \phi$, i.e. there exists $v \in W^L_{can}$ such that $\|\phi\|_{M_{can},v} = v(\phi^*) < 1$. □

As in the classical case, the logics $K45(Ł_k)$, $KD45(Ł_k)$ and $S5(Ł_k)$ admit simpler semantics while preserving sound- and completeness.

Consider the following classes of Kripke frames:

- $\mathbb{M}^s_{\text{et}}$: the subclass of Kripke frames $M = (W, e, R)$ of $\mathbb{M}_{\text{et}}$ where $R = W \times E$ for some $E \subseteq W$
- $\mathbb{M}^s_{\text{est}}$: the subclass of Kripke frames $M = (W, e, R)$ of $\mathbb{M}_{\text{est}}$ where $R = W \times E$ for some $\emptyset \neq E \subseteq W$
- $\mathbb{M}^s_{\text{rst}}$: the subclass of Kripke frames $M = (W, e, R)$ of $\mathbb{M}_{\text{rst}}$ where $R = W \times W$

Proposition 3.5. *$K45(Ł_k)$, $KD45(Ł_k)$ and $S5(Ł_k)$ are sound and complete w.r.t. the classes $\mathbb{M}^s_{\text{et}}$, $\mathbb{M}^s_{\text{est}}$ and $\mathbb{M}^s_{\text{rst}}$ respectively.*

Proof. We only prove the case of $KD45(Ł_k)$, the other cases being easy variations. By Theorem 3.4, it is sufficient to show that $\mathbb{M}_{\text{est}}$ and $\mathbb{M}^s_{\text{est}}$ have the same tautologies. Since $\mathbb{M}^s_{\text{est}}$ is a subclass of $\mathbb{M}_{\text{est}}$, we only have to show that if for a formula ϕ there exists $M = (W, e, R) \in \mathbb{M}_{\text{est}}$ and $w \in W$ such that $\|\phi\|_{M,w} < 1$, that there exists $M' = (W', e', R') \in \mathbb{M}^s_{\text{est}}$ and $w' \in W'$ such that $\|\phi\|_{M',w'} < 1$. Suppose such a frame $M = (W, e, R) \in \mathbb{M}_{\text{est}}$ and $w \in W$ are given. Define $E = \{v \in W \mid R(w, v) = 1\}$. By seriality of R we have $E \neq \emptyset$. We define M' as follows: $W' = \{w\} \cup E$, $e' = e_{|W' \times V}$ and $R' = W' \times E$. Notice that for $v \in E$ arbitrary we have $E \subseteq \{z \mid R(v, z) = 1\}$ since R is Euclidean and $\{z \mid R(v, z) = 1\} \subseteq E$ since R is transitive. Hence $E = \{z \mid R(v, z) = 1\}$ for all $v \in E$. We can now show by structural induction that for each ψ it holds that $\|\psi\|_{M,v} = \|\psi\|_{M',v}$ for every $v \in E$. The only notable case is when $\psi = \mathrm{B}\alpha$, but this follows by the previous remark: $\|\mathrm{B}\alpha\|_{M,v} = \inf\{\|\alpha\|_{M,v'} \mid v' \in E\} = \|\mathrm{B}\alpha\|_{M',v}$. Finally, it is sufficient to show that $\|\phi\|_{M,w} = \|\phi\|_{M',w}$. We do this by structural induction, again we only show the case $\phi = \mathrm{B}\psi$: $\|B\psi\|_{M,w} = \inf\{\|\psi\|_{M,v} \mid v \in E\} = \inf\{\|\psi\|_{M',v} \mid v \in E\} = \|\mathrm{B}\psi\|_{M',w}$. □

4 Complexity of satisfiability problems

In this section we will discuss the complexity of two satisfiability problems for $KD45(Ł_k)$.

- 1-SAT: Given a formula ϕ, does there exist $M = (W, e, R) \in \mathbb{M}^s_{\text{est}}$ and $w \in W$ such that $\|\phi\|_{M,w} = 1$?
- *pos*-SAT: Given a formula ϕ, does there exist $M = (W, e, R) \in \mathbb{M}^s_{\text{est}}$ and $w \in W$ such that $\|\phi\|_{M,w} > 0$?

We will show that these problems are NP-complete and hence generalizing to the many-valued case does not imply an increase in computational complexity. See [Halpern and Moses, 1992] for results on the complexity of classical modal logics. As in the previous section, the same results can be obtained for $K45(Ł_k)$ and $S5(Ł_k)$.

For any formula ϕ of $\mathcal{L}_{\mathcal{B}}$, we denote by $\#\phi$ its complexity:

- $\#\overline{c} = 1$ for each $c \in S_k$ and $\#p = 1$ for every propositional variable p
- $\#(\phi \rightarrow \psi) = 1 + \#\phi + \#\psi$ and $\#(\mathrm{B}\phi) = 1 + \#\phi$.

For a formula ϕ of $KD45(Ł_k)$ and a subformula ψ of ϕ, the *depth* $d(\psi)$ of ψ in ϕ is defined as usual given the tree of subformulas of ϕ. For instance, for a formula $\mathrm{B}(a \wedge \mathrm{B}b)$, we have $d(\mathrm{B}(a \wedge \mathrm{B}b)) = 0$, $d(a \wedge \mathrm{B}b) = 1$, $d(a) = d(\mathrm{B}b) = 2$ and $d(b) = 3$. We can then show the following finite model property:

Lemma 4.1. *Let ϕ be a $\mathcal{L}_{\mathcal{B}}$-formula. Then for every frame $M=(W,e,R)\in\mathbb{M}^s_{\text{est}}$, and for every $w\in W$, there exists a finite frame $M'=(W',e',R')\in\mathbb{M}^s_{\text{est}}$ and a world $w'\in W'$ such that $|W'|\leq\#\phi$ and $\|\phi\|_{M,w}=\|\phi\|_{M',w'}$.*

Proof. Consider a frame $M=(W,e,R)$ with $R=W\times E$ and $w\in W$. The aim is to find a finite set W' and a non empty subset $E'\subseteq W'$ for which the claim holds.

Trivially, if ϕ is B-free, then take $W'=E'=\{w\}$, $R'=W'\times E'$ and let e' be defined by restriction.

Otherwise, if ϕ is not B-free, let d be the maximum depth of the subformulas of ϕ of the form $\mathrm{B}\psi$.

If $d=0$, then $\phi=\mathrm{B}\psi$ and ψ is B-free. Then $\|\mathrm{B}\psi\|_{M,w}=\inf\{\|\psi\|_{M,w^*}\mid w^*\in E\}$. Now, since $\|\psi\|_{M,w^*}=e(w^*,\psi)$ can only take a finite number of values in S_k, there exists a world $w^0\in W$ in which the infimum is attained, i.e. $\|\mathrm{B}\psi\|_{M,w}$ equals

$$\inf\{\|\psi\|_{M,w^*}\mid w^*\in E\}=\|\psi\|_{M,w^0}=e(w^0,\psi).$$

In this case put $W'=\{w^0\}$, $w'=w^0$ and let E' and e' be defined by restriction.

If $d>0$, let $\mathrm{B}\psi_{d1},\dots,\mathrm{B}\psi_{dr_d}$ be the subformulas of ϕ of depth d, hence each ψ_{dj} is B-free. Again, for each ψ_{dj}, there exists a world w^{dj} such that

$$\|\mathrm{B}\psi_{dj}\|_{M,w^*}=\|\psi_{dj}\|_{M,w^{dj}}=e(w^{dj},\psi_{dj})$$

with $w*$ arbitrary. Now replace each subformula $\mathrm{B}\psi_{dj}$ by the corresponding constant and repeat the process for all levels $n=d-1,\dots,0$. Put $W'=\{w\}\cup\{w^{lj}\mid 1\leq l\leq d, 1\leq j\leq r_l\}$, $w'=w$ and let E' and e' be defined by restriction to W'. Then, by construction, $\|\phi\|_{M,w}=\|\phi\|_{M',w'}$. Moreover, $|W'|=1+\sum_{l=0}^{d}r_l\leq\#\phi$. □

Observe that, as in the proof of Lemma 4.1, given a formula ϕ of $KD45(\text{Ł}_k)$ and a frame $M=(W,e,R)$, we can construct an Ł_k formula ϕ^M.

Theorem 4.2. *The problems 1-SAT and pos-SAT for $KD45(\text{Ł}_k)$ are NP-complete.*

Proof. In order to prove NP-membership, recall that from Lemma 4.1 a formula ϕ is 1-SAT in a frame M iff ϕ is 1-SAT in a finite frame M' whose cardinality is polynomial in the complexity of ϕ. Let us guess the frame $M'=(W',e',R')$. Since $|W'|\leq\#\phi$, the guess is polynomial in $\#\phi$. Let $\phi^{M'}$ the formula of Ł_k obtained from M' and ϕ applying the procedure described above, and notice that $\#\phi^{M'}$ is polynomial in $\#\phi$. Moreover, since $|W'|\leq\#\phi$ the formula $\#\phi^{M'}$ is obtained in a number of steps which is polynomial in $\#\phi$. From [Mundici, 1987] it follows that checking if $\phi^{M'}$ is either 1-SAT or *pos*-SAT in Ł_k is NP, and hence our claim follows. Since each formula of Ł_k is in particular a formula of $KD45(\text{Ł}_k)$, and since both 1-SAT and *pos*-SAT for Ł_k are NP-complete, the NP-hardness of our problems follows. □

Notice that in [Bou *et al.*, 2011a], the authors considered these satisfiability problems for the minimal modal logic over Ł_k, but with respect to generic Kripke frames where the accessibility relation can also be many-valued, and they proved that those problems are PSPACE-complete.

5 Relating fuzzy modal logic and fuzzy autoepistemic logic

The aim of Moore's autoepistemic logic [Moore, 1983] is to model or characterize the set of beliefs of a rational agent with introspection capabilities by means of a set of simple properties it must fulfill. The language of fuzzy autoepistemic logic is the same as $\mathcal{L}_{\mathcal{B}}$, the one of the modal logic with the belief operator B, so $\mathrm{B}\varphi$ is also read as "the agent believes φ". The basic construct is the notion *stable expansion* E of a set of initial beliefs A, briefly introduced in Section 1. It can be seen as the closed set of beliefs of an ideal rational agent reflecting on his own beliefs. This notion has been generalized to the fuzzy case in [Blondeel *et al.*, 2013] as follows. A *stable fuzzy expansion* of a set of $\mathcal{L}_{\mathcal{B}}$-formulas A is a fuzzy set $E_A:\mathcal{L}_{\mathcal{B}}\to S_k$ that satisfies the following fix-point condition:

$$E_A(\phi)=\inf\{v(\phi^*)\mid v\in\Omega_k,\\ v(\alpha)=1,\forall\alpha\in A^*\cup\{(\mathrm{B}\psi)^*\leftrightarrow\overline{E_A(\psi)}\mid\psi\in\mathcal{L}_{\mathcal{B}}\}\}.$$

Recall that $\overline{E_A(\psi)}$ is the symbol corresponding to $E_A(\psi)\in S_k$.

Generalizing Moore's result [Moore, 1984], in [Blondeel *et al.*, 2013] it is shown that stable fuzzy expansions can also be characterized by a fuzzy Kripke-style possible world semantics. In particular, a *fuzzy autoepistemic structure* is a pair (w,S) with $w\in\Omega_k$ representing the actual world used to evaluate B-free formulas and $S\subseteq\Omega_k$ representing all worlds considered possible (epistemic states) used to evaluate formulas of the form $\mathrm{B}\psi$. The class of such structures will be denoted by $\mathbb{M}^{\text{ae}}$. The degree of truth of a $\mathcal{L}_{\mathcal{B}}$-formula ϕ relative to a fuzzy autoepistemic structure (w,S) is inductively defined as follows:

- If ϕ is a propositional formula from $\mathcal{L}$, then $\|\phi\|_{(w,S)}=w(\phi)$.
- If ϕ is a truth constant $\overline{c}$, then $\|\phi\|_{(w,S)}=c$.
- If $\phi=\mathrm{B}\psi$, then $\|\mathrm{B}\psi\|_{(w,S)}=\inf\{\|\psi\|_{(v,S)}\mid v\in S\}$.
- if $\phi=\psi\to\gamma$, $\|\phi\|_{(w,S)}=(\|\psi\|_{(w,S)}\Rightarrow\|\gamma\|_{(w,S)})$.

Intuitively, one can think of S as a set of "sources" (worlds) and we define the truth value of $\mathrm{B}\varphi$ in S as the minimal value of φ such that each source supports it at least in this degree. Since the truth evaluation of formulas of the form $\mathrm{B}\varphi$ in a structure (w,S) does not depend on the actual wold w, we will also write $\|\mathrm{B}\varphi\|_S$ to denote $\|\mathrm{B}\varphi\|_{(w,S)}$. Note that if $S=\emptyset$, then $\|\mathrm{B}\varphi\|_S=1$. Also note that, conversely, the world w in (w,S) is needed to evaluate non-modal formulas.
We consider the following subclasses of fuzzy autoepistemic structures of $\mathbb{M}^{\text{ae}}$:

- the class $\mathbb{M}^{\text{ae}}_e$, where only pairs (w,S) with S non-empty are considered, and
- the class $\mathbb{M}^{\text{ae}}_{in}\subseteq\mathbb{M}^{\text{ae}}_e$, where only pairs (w,S) with $w\in S$ are considered.

It can be shown that $K45(\text{Ł}_k)$, $KD45(\text{Ł}_k)$ and $S5(\text{Ł}_k)$ are still sound and complete with respect to the classes $\mathbb{M}^{\text{ae}}$, $\mathbb{M}^{\text{ae}}_e$ and $\mathbb{M}^{\text{ae}}_{in}$ respectively.

Theorem 5.1. *$K45(\text{Ł}_k)$, $KD45(\text{Ł}_k)$ and $S5(\text{Ł}_k)$ are sound and complete w.r.t. to $\mathbb{M}^{\text{ae}}$, $\mathbb{M}^{\text{ae}}_e$ and $\mathbb{M}^{\text{ae}}_{in}$, respectively.*

Proof. We only show the case of $KD45(Ł_k)$. The other cases are obtained by slight adaptions of the proof. By Proposition 3.5 it is sufficient to show that $\mathbb{M}^s_{\text{est}}$ and $\mathbb{M}^{\text{ae}}_e$ have the same tautologies. First suppose there exists $M = (W, e, R) \in \mathbb{M}^s_{\text{est}}$ and $w \in W$ such that $\|\phi\|_{M,w} < 1$. Define $(w, S) \in \mathbb{M}^{\text{ae}}_e$ such that $R = W \times S$. We can then show by structural induction that for each formula γ we have $\|\gamma\|_{M,v} = \|\gamma\|_{(v,S)}$ for all $v \in W$. Next, suppose we have $(w, S) \in \mathbb{M}^{\text{ae}}_e$ such that $\|\phi\|_{(w,S)} < 1$. Define $M = (W, e, R) \in \mathbb{M}^s_{\text{est}}$ as follows: $W = \{w\} \cup S$ and $R = W \times S$. Similar as above it follows that $\|\gamma\|_{M,w} = \|\gamma\|_{(w,S)}$ for each formula γ. □

In [Blondeel *et al.*, 2013] the authors characterize stable fuzzy expansions in terms of this possible world many-valued semantics. Indeed, the *fuzzy belief set Bel_S* induced by an epistemic state described by a set of $Ł_k$-evaluations S is defined in the natural way: for each $\mathcal{L_B}$-formula φ

$$Bel_S(\varphi) = \|\mathrm{B}\varphi\|_S = \inf_{w \in S} \|\varphi\|_{(w,S)}.$$

This notion also generalizes that of a S5-set to the many-valued case. Moreover, given a set of formulas A, a set of propositional evaluations S_A is called a fuzzy autoepistemic model of A whenever

$$S_A = \{w \in \Omega_k \mid \|\phi\|_{(w,S_A)} = 1 \text{ for each } \phi \in A\}.$$

Intuitively, S_A is the set of worlds or "sources" in which all formulas of A are true. Then the following result is proved in [Blondeel *et al.*, 2013].

Proposition 5.2. *A fuzzy set of formulas $E : \mathcal{L}_B \to S_k$ is a stable fuzzy expansion of a set of formulas A iff it is the belief set for some fuzzy autoepistemic model S_A of A, i.e. $E(\phi) = \|\mathrm{B}\phi\|_{S_A}$ for each $\phi \in \mathcal{L_B}$.*

On the other hand, as it happens in the classical case, we can also characterize fuzzy belief sets, or equivalent stable fuzzy expansions, by means of the syntactical notion of fuzzy stable sets (c.f. [Halpern and Moses, 1992]).

Definition 5.3. Let $\Gamma : \mathcal{L_B} \to S_k$ be a fuzzy set and put $\hat{\Gamma} = \{\overline{\Gamma(\varphi)} \to \varphi^* \mid \varphi \text{ formula}\}$. We say that Γ is a *fuzzy stable set* if the following conditions hold:

(1) $\hat{\Gamma}$ is propositionally consistent, i.e. $\hat{\Gamma} \not\vdash \overline{0}$.

(2) If $\hat{\Gamma} \vdash \bar{c} \to \varphi^*$ then $\Gamma(\varphi) \geq c$.

(3) $\Gamma(\varphi) = \Gamma(\mathrm{B}\varphi)$

(4) $1 - \Gamma(\varphi) = \Gamma(\neg\mathrm{B}\varphi)$

Proposition 5.4. Γ *is a fuzzy stable set iff* Γ *is a fuzzy belief set.*

Proof. (1) First we show that a fuzzy belief set Γ is a fuzzy stable set. By definition of a fuzzy belief set we know that there exists $S \subseteq \Omega_k$ such that $\Gamma(\varphi) = \|\mathrm{B}\varphi\|_S$ for each formula φ. In order to show that $\hat{\Gamma}$ is propositionally consistent, by the strong completeness of $Ł_k$, it is sufficient to show that there exists $v \in \Omega_k$ such that for each formula φ we have $\Gamma(\varphi) \leq v(\varphi^*)$. Let $w \in S$ be arbitrary but fixed and define v such that $v(\varphi^*) = \|\varphi\|_{(w,S)}$ for each φ. It follows that $\Gamma(\varphi) \leq v(\varphi^*)$ which proves (1). Next, assume that $\hat{\Gamma} \vdash \bar{c} \to \varphi^*$, or by strong completeness of $Ł_k$ that $\hat{\Gamma} \models \bar{c} \to \varphi^*$. We show that $\Gamma(\varphi) \geq c$. Note, similar as above, that for each $w \in S$ we have that v with $v(\varphi^*) = \|\varphi\|_{(w,S)}$ is a model of $\hat{\Gamma}$ and hence of $\bar{c} \to \varphi^*$. Therefore $c \leq \|\varphi\|_{(w,S)}$ for each $w \in S$ and $c \leq \inf_{w\in S} \|\varphi\|_{(w,S)} = \Gamma(\varphi)$. Proving (3) follows easily by noting that $\Gamma(\varphi) = \|\mathrm{B}\varphi\|_S = \|\mathrm{BB}\varphi\|_S = \Gamma(\mathrm{B}\varphi)$. Finally, to show (4), observe that $\Gamma(\neg\mathrm{B}\varphi) = \|\mathrm{B}\neg\mathrm{B}\varphi\|_S = \inf_{w\in S} \|\neg\mathrm{B}\varphi\|_{(w,S)} = \inf_{w\in S}(1 - \|\mathrm{B}\varphi\|_{(w,S)}) = \inf_{w\in S}(1 - \Gamma(\varphi)) = 1 - \Gamma(\varphi)$.

(2) Now, let Γ be fuzzy stable set. Define $S = \{u \in \Omega_k \mid u(\alpha) = 1, \forall\alpha \in \hat{\Gamma}\}$. Note that S is nonempty by (1). Next, note that for each $w \in S$ we have by (3) that $w((\mathrm{B}\varphi)^*) \geq \Gamma(\mathrm{B}\varphi) = \Gamma(\varphi)$. We show that also $w((\mathrm{B}\varphi)^*) \leq \Gamma(\varphi)$ from which it then follows that $w((\mathrm{B}\varphi)^*) = \Gamma(\varphi)$ for each $w \in S$. Indeed, by (4) we have that $\Gamma(\varphi) = 1 - \Gamma(\neg\mathrm{B}\varphi)$ and since $\Gamma(\neg\mathrm{B}\varphi) \leq w((\neg\mathrm{B}\varphi)^*) = w(\neg(\mathrm{B}\varphi)^*)$, the latter is greater or equal than $1 - w(\neg(\mathrm{B}\varphi)^*) = w((\mathrm{B}\varphi)^*)$. We will now show that for each formula α and each $w \in S$ we have $w(\alpha^*) = \|\alpha\|_S$ from which we can conclude that $\Gamma(\varphi) = w((\mathrm{B}\varphi)^*) = \|\mathrm{B}\varphi\|_S$ for each formula φ and an arbitrary $w \in S \neq \emptyset$. The only notable case is where $\alpha = \mathrm{B}\psi$. By the induction hypothesis and by the definition of S we have $\|\mathrm{B}\psi\|_S = \inf_{v\in S} \|\psi\|_{(v,S)} = \inf_{v\in S} v(\psi^*) \geq \Gamma(\psi)$. Now suppose that for each $w \in S$ we have $w(\psi^*) > \Gamma(\psi)$. Since the set of truth values is finite there exists $w' \in S$ such that $w'(\psi^*) = \min_{w\in S} w(\psi^*)$ and hence $\hat{\Gamma} \vdash \overline{w'(\psi^*)} \to \psi^*$. By (2), it then follows that $\Gamma(\psi) \geq w'(\psi^*)$ and hence that $\Gamma(\psi) > \Gamma(\psi)$, a contradiction. □

The results from this section show that the modal logics we are dealing with are suitable for reasoning in the fuzzy autoepistemic frame. In particular, the following example illustrates that real-world situations, which can be naturally expressed in the autoepistemic language, can also be framed in our fuzzy modal setting.

Example 5.5. Consider the following example to show how fuzzy autoepistemic logic can be used in a real world scenario. Suppose we have a wireless sensor network consisting of devices that can sense their environment and communicate wirelessly, for instance with the purpose of detecting forest fires in an early stage. We will use fuzzy autoepistemic logic to determine whether there are sensors not working optimally and if so, within what range we can assume the temperature to be.

Let t_i be the actual temperature at sensor i and t'_i the temperature measured by sensor i. Suppose we have an appropriate rescaling to assure that all variables take values in Ω_k and let e_i be the variable representing the degree to which sensor i is faulty. The formula

$$\neg\mathrm{B}e_i \to (t_i \leftrightarrow t'_i) \tag{1}$$

then captures a fuzzy version of the classical intuition that "If we do not believe that the sensor i is broken, then it measures the right temperature". Moreover, let us define a new connective $d(\varphi, \psi)$[3] as $\neg(\varphi \leftrightarrow \psi)$ for which the semantics

[3] The connective $d(\varphi, \psi)$ is well known in the literature of many-

is given by the Euclidean distance $\hat{d}$: for all $x, y \in [0,1]$, $\hat{d}(x,y) = |x-y|$. The formula

$$(w_{ij} \to d(t'_i, t'_j)) \to e_i \vee e_j \tag{2}$$

then has the following intuitive meaning: "If the difference between the measurements of sensors i and j is above some threshold w_{ij} (to a certain degree), then there must be something wrong with one of the sensors. This value w_{ij} can for instance be based on the location of the sensors i and j.

As a concrete example, suppose we have 2 sensors such that the temperature measured by sensor 1 is equal to 0.4 and by sensor 2 equal to 0.5. Suppose the threshold is equal to 0.2. For a fuzzy autoepistemic model S of formulas (1) and (2), it must then hold for each $w' \in S$ that

(a) $0.4 - \inf_{w \in S} w(e_1) \leq w'(t_1) \leq 0.4 + \inf_{w \in S} w(e_1)$

(b) $0.5 - \inf_{w \in S} w(e_2) \leq w'(t_2) \leq 0.5 + \inf_{w \in S} w(e_2)$

(c) $0.9 \leq \max(w'(e_1), w'(e_2))$

where (a) and (b) follow from the fact that (w', S) must satisfy equation (1), and (c) from the fact that it must satisfy equation (2). One can easily show that $S = \{w \in \Omega_k \mid 0.9 \leq \max(w(e_1), w(e_2)), w(t_1) = 0.4, w(t_2) = 0.5\}$ is a fuzzy autoepistemic model of formulas (1) and (2). In the corresponding stable fuzzy expansion E it holds $E(t_1) = 0.4$, $E(t_2) = 0.5$, $E(e_1) = E(e_2) = 0$ and $E(e_1 \vee e_2) = 0.9$. Hence we can conclude that there is no error made by the sensors.

6 "Only knowing" operators and fuzzy stable expansions

Let us expand $\mathcal{L}_{\mathrm{B}}$ with a modal operator O such that a formula $\mathrm{O}\psi$ has to be interpreted as "ψ is all that is believed". In the classical case [Levesque, 1990], the semantics is defined as follows. Given a epistemic state S consisting of a set of classical evaluations, a formulas $\mathrm{O}\psi$ is true in S when ψ is true in any structure (z, S) with $z \in S$, and false in any structure (z', S) with $z' \notin S$.

We can straightforwardly generalize this condition to the many-valued case by defining

$$\|\mathrm{O}\psi\|_{(w,S)} = \min(\inf_{z \in S} \|\psi\|_{(z,S)}, \inf_{z \notin S} \|\neg\psi\|_{(z,S)}),$$

where now $w \in \Omega_k$ and $S \subseteq \Omega_k$. Formulas of the form $\mathrm{B}\psi$ are evaluated as in fuzzy autoepistemic logic. If we then add another modal operator N whose truth evaluation in a pair (w, S) is

$$\|\mathrm{N}\psi\|_{(w,S)} = \inf_{z \notin S} \|\psi\|_{(z,S)},$$

then it is easy to see that the semantics of $\mathrm{O}\psi$ is exactly that of $\mathrm{B}\psi \wedge \mathrm{N}\neg\psi$. Notice that $\|\mathrm{N}\psi\|_{(w,S)} = \|\mathrm{B}\psi\|_{(w,\Omega_k \setminus S)}$, so it is clear that N is another K45 operator. Again, since the truth value of $\mathrm{O}\psi$ (and $\mathrm{N}\psi$) in a structure (w, S) does not depend on w, we will also write $\|\mathrm{O}\psi\|_S$ and $\|N\psi\|_S$ to denote $\|\mathrm{O}\psi\|_{(w,S)}$ and $\|N\psi\|_{(w,S)}$ respectively.

valued logics and it is usually called *Chang distance function* [Cignoli *et al.*, 2000]. The fact that d can be defined in a many-valued logical setting is a peculiarity of MV-algebras and also for this reason we believe these structures to be a suitable algebraic setting.

In [Levesque, 1990], Levesque proposes a sound and complete axiomatization for the classical logic of "only knowing". It is worth noticing that all axioms are also valid in our fuzzy framework. We will check in particular the ones involving both operators:

Lemma 6.1. *The following formula schemas are valid:*

(i) $\phi \to \mathrm{B}\phi$*, where all variables and constants in* ϕ *occur in the scope of an operator* N *or* B*,*

(ii) $\phi \to \mathrm{N}\phi$*, where all variables and constants in* ϕ *occur in the scope of an operator* N *or* B*,*

(iii) $\neg\mathrm{B}\phi \vee \neg\mathrm{N}\phi$*, if* $\neg\phi$ *is satisfiable and does not contain any modal operators.*

Proof. Schemas (i) and (ii) are easy to check. For condition (iii), suppose $\neg\phi$ is satisfiable, i.e. there exists a structure (w', S') such that $w'(\phi) = \|\phi\|_{(w',S')} = 0$. For a structure (w, S) we have $\|\neg\mathrm{B}\phi \vee \neg\mathrm{N}\phi\|_{(w,S)} = 1$ iff $\max(\|\neg\mathrm{B}\phi\|_{(w,S)}, \|\neg\mathrm{N}\phi\|_{(w,S)}) = 1$ iff $\|\mathrm{B}\phi\|_{(w,S)} = 0$ or $\|\mathrm{N}\phi\|_{(w,S)} = 0$ iff there exists $z \in S$ such that $z(\phi) = \|\phi\|_{(z,S)} = 0$ or there exists $z \notin S$ such that $z(\phi) = \|\phi\|_{(z,S)} = 0$. The latter is satisfied by the fact that there exists $w' \in \Omega_k$ such that $w'(\phi) = 0$. $\square$

Notice that (i) and (ii) are more general than both modal axioms (4) and (5) (see Section 3). The question of whether axioms $(i), (ii)$ and (iii), together with the minimal modal logic axioms for B and for N, provide a complete axiomatization with respect to the above semantics as well as its complexity is left as future work. We can only advance now that the logic is decidable since a re-adaptation of Lemma 4.1 easily shows that the "only knowing" logic over finitely-valued Lukasiewicz logic has the finite model property. Nothing about the complexity can be stated up to now, since no polynomial bound on the cardinality of the model has been fixed so far.

Nonetheless, we can show that the relationship between the "only knowing" operator O and Moore's stable expansions proved in [Levesque, 1990] nicely extends to our fuzzy framework with some slight variations. First of all, observe that if a formula $\mathrm{O}\varphi$ gets value 1 in an epistemic state S, then necessarily φ must be Boolean.

Lemma 6.2. *For any formula* φ*, if* $\|\mathrm{O}\varphi\|_S = 1$ *then* $\|\varphi\|_{(v,S)} = 1$ *for every* $v \in S$ *and* $\|\varphi\|_{(v',S)} = 0$ *for every* $v' \notin S$*, hence* φ *is Boolean.*

Next proposition shows that the belief set Bel_S for an epistemic state defined by a set of Ł$_k$-evaluations S is indeed a stable fuzzy expansion of a Boolean premise φ whenever φ is all what is fully believed in the epistemic state S.

Proposition 6.3. *For any Boolean* $\mathcal{L}_{\mathcal{B}}$*-formula* φ*,* $\|\mathrm{O}\varphi\|_S = 1$ *iff* Bel_S *is a stable fuzzy expansion of* $A = \{\varphi\}$*.*

Proof. Since $\|\mathrm{O}\varphi\|_S = \min(\|\mathrm{B}\varphi\|_S, \|\mathrm{N}\neg\varphi\|_S)$, we have the following chain of equivalences:

$$
\begin{array}{lll}
\|O\varphi\|_S = 1 & \text{iff} & \|B\varphi\|_S = 1 \text{ and } \|N\neg\varphi\|_S = 1 \\
& \text{iff} & \forall v \in S,\ \|\varphi\|_{(v,S)} = 1 \text{ and} \\
& & \forall v \notin S,\ \|\varphi\|_{(v,S)} = 0 \\
& \text{iff} & S = \{v \in \Omega_k \mid \|\varphi\|_{(v,S)} = 1\} \\
& \text{iff} & S \text{ is a fuzzy AE model of } A \\
& \text{iff} & Bel_S \text{ is a stable fuzzy expansion of } A.
\end{array}
$$

where the last equivalence follows from Prop. 5.2. Notice that the third "iff" is valid only in case φ is Boolean. □

7 Related work and conclusions

In this paper we have introduced Hilbert-style axiomatizations of fuzzy modal logics for belief based on finitely-valued Łukasiewicz logic, in particular of many-valued counterparts of the well-known K45, KD45 and S5 modal logics. We have shown they provide possible-world semantics for a fuzzy autoepistemic logic, generalizing some bridges established in the classical case in [Levesque, 1990]. In the last years there has been some work on fuzzy modal logics with generalized Kripke semantics, see e.g. [Bou *et al.*, 2011b] and references therein. Here we have focused on modal systems based on a finite set of linearly ordered truth-values with Łukasiewic logic semantics for connectives which generate the class of finite MV-algebras [Mundici, 1987]. These systems represent a good compromise between expressive power and nice logical properties. The infinitely-valued case offers some problems, see e.g. [Hájek, 2010] for the case of S5 with total accessibility relations. On the other hand, a closely related work is Maruyama's paper [Maruyama, 2011], where modal logics for belief based on a finitely-valued Heyting algebra of truth-values and hence not necessarily linearly ordered are considered. The formalization is very similar, but he also deals with common belief. Here we rather focus on providing a formal basis for the fuzzy generalization of autoepistemic logic developed in [Blondeel *et al.*, 2013]. Many-valued extensions of autoepistemic logic have also been addressed by Fitting [Fitting, 1992] in the context of finite Heyting algebras of truth-values, as well as by Koutras et al. [Koutras and Zachos, 2000; Koutras *et al.*, 1999]. As for future work, we plan to fully develop the "only believing" logic with the two modal operators O and N, and extend the modal approach to possibly other fuzzy logics.

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